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Overview of Robust Design, Propagation of Error and Tolerance Analysis

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Making the most of this learning opportunity

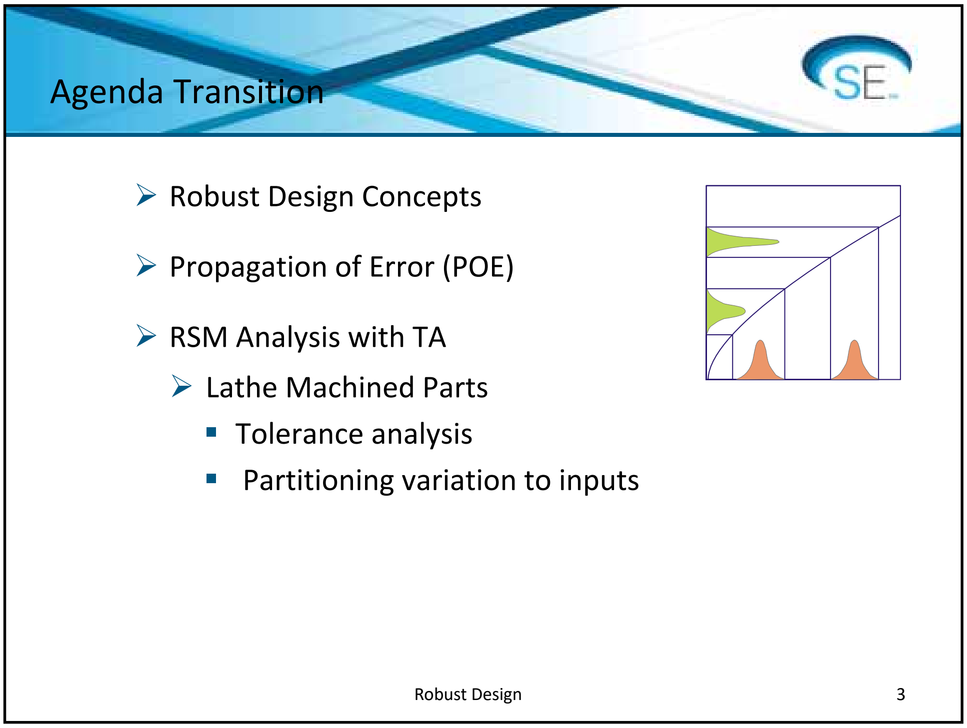
Hide the Control Panel
Mute your line

To prevent audio disruptions, all attendees will be muted.

Questions can be posted in the Question area. If they are not addressed during the webinar, I will reply via email afterwards.

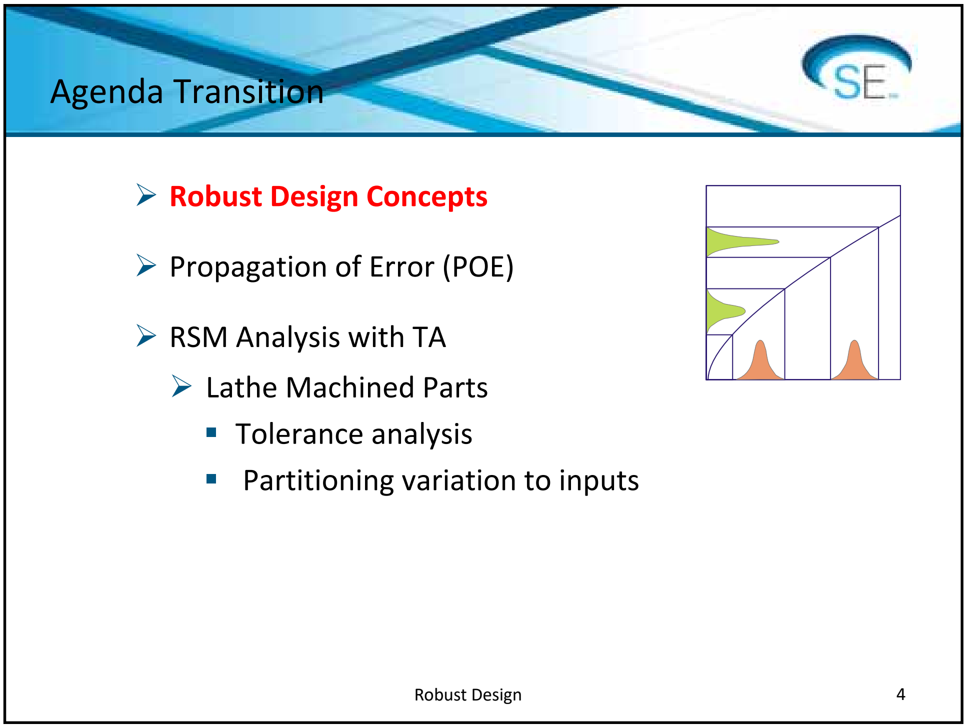
Questions may also be sent to stathelp@statease.com. Please provide your company name and, if you are using Design-Expert, the serial number (found under Help, About).

Note: The slides and a recording of this webinar will be posted on the Webinars page of the Stat-Ease website within a few days.



Agenda Transition

- Robust Design Concepts
- Propagation of Error (POE)
- RSM Analysis with TA
  - Lathe Machined Parts
    - Tolerance analysis
    - Partitioning variation to inputs



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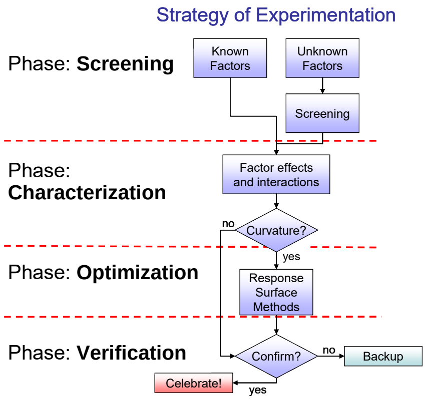
Robust Design

Functional Design:

- Use DOE to model response **mean** as a function of controllable factor levels.
- Choose levels of controllable factors to achieve targeted values of the responses.



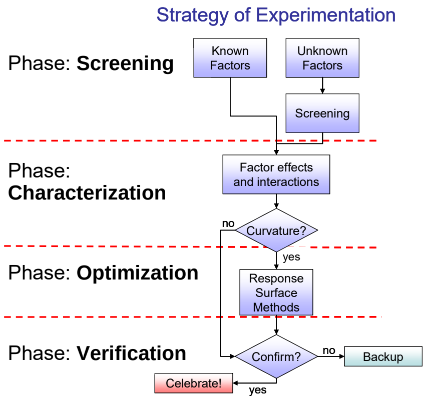
Strategy of Experimentation

Phase: **Screening**
Known Factors
Unknown Factors
Screening
Phase: **Characterization**
Factor effects and interactions
no
Curvature?
yes
Phase: **Optimization**
Response Surface Methods
Phase: **Verification**
Confirm?
no
Backup
Celebrate!
yes

Robust Design

Robust Design:

- Use DOE to model response **variability** as a function of control and uncontrolled factor levels.
- Choose levels of control factors to reduce variation caused by:
  - Lack of control of the control factors.
  - Variation of the uncontrolled factors.



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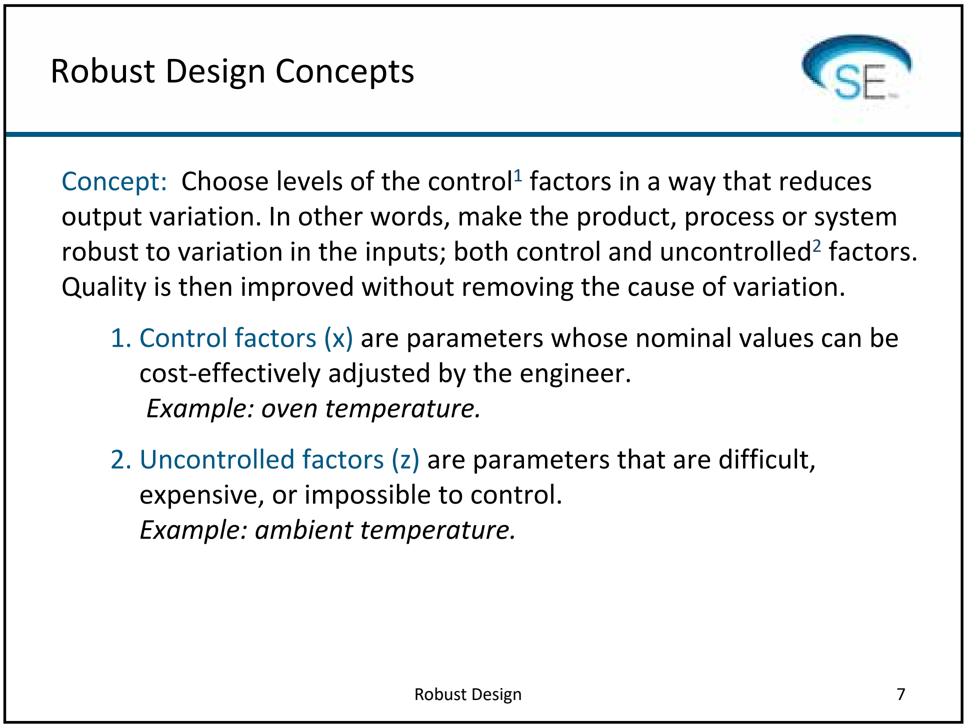
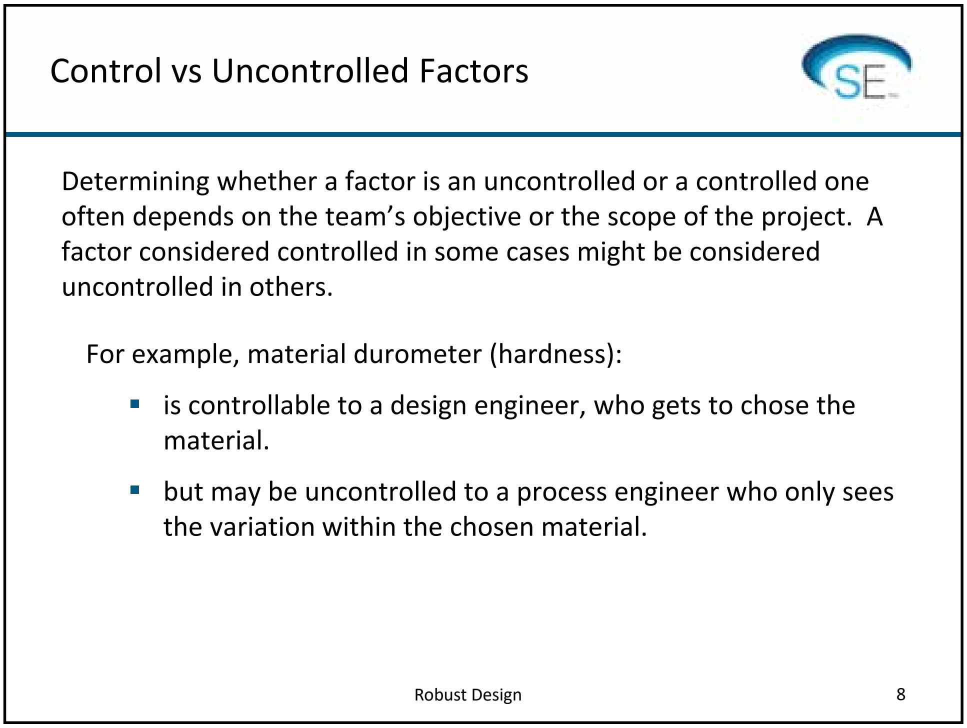


Robust Design Concepts

Concept: Choose levels of the control[1] factors in a way that reduces output variation. In other words, make the product, process or system robust to variation in the inputs; both control and uncontrolled[2] factors. Quality is then improved without removing the cause of variation.

1. Control factors (x) are parameters whose nominal values can be cost-effectively adjusted by the engineer.
   Example: oven temperature.
2. Uncontrolled factors (z) are parameters that are difficult, expensive, or impossible to control.
   Example: ambient temperature.



Control vs Uncontrolled Factors

Determining whether a factor is an uncontrolled or a controlled one often depends on the team’s objective or the scope of the project. A factor considered controlled in some cases might be considered uncontrolled in others.

For example, material durometer (hardness):

- is controllable to a design engineer, who gets to chose the material.
- but may be uncontrolled to a process engineer who only sees the variation within the chosen material.

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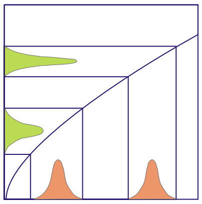
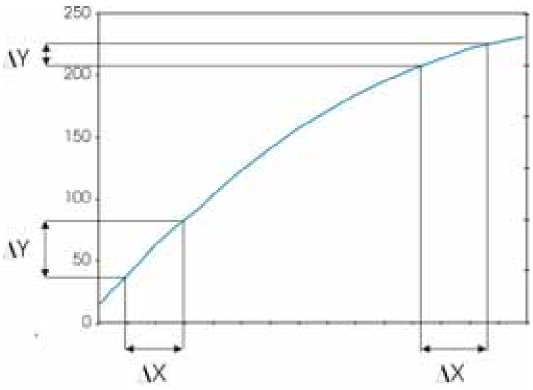


Propagation of Error (POE)

Transmitted Variation

Objective: Reduce the variation transmitted to the response from variation in control factors.



Propagation of Error

How it works

Once a relationship has been established between a factor and a response, the variation in the output can be:

1. Dependent on the level of the control factor
2. Independent of the level of the control factor

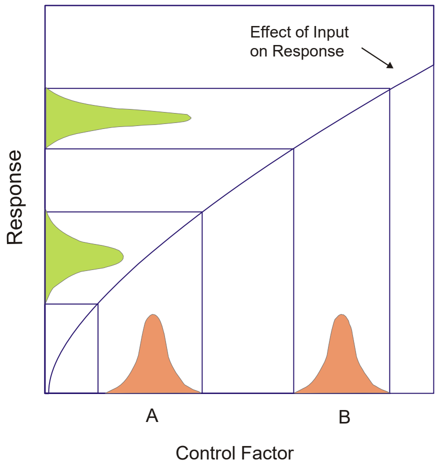
See pictures on next two pages →

Propagation of error

Dependent

The transmitted variation is **dependent** on the level of the control factor.

Therefore, set the level of the control factor to reduce variation transmitted to the response from variation (lack-of-control) of the control factor.

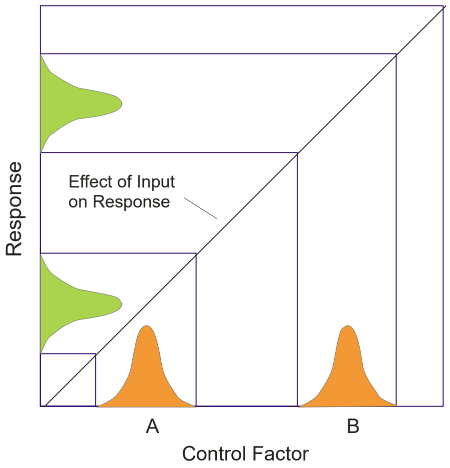


Propagation of error

Independent

The transmitted variation is **independent** of the level of the control factor.

Therefore, set the level of the control factor to center the process mean on target.

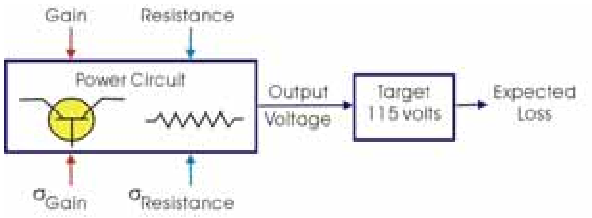


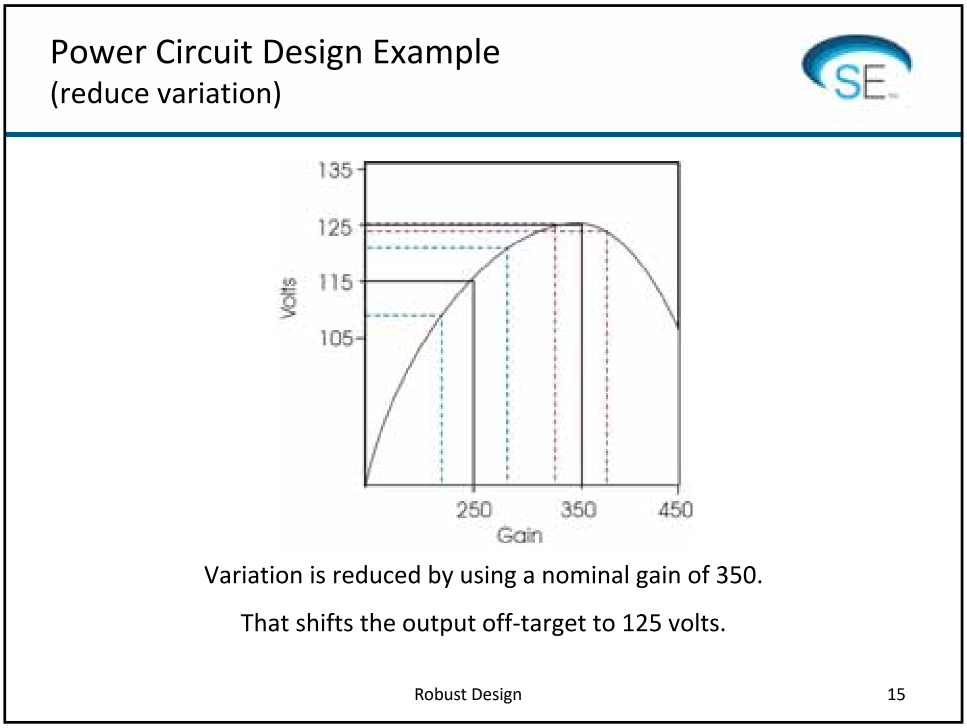
Power Circuit Design Example

Consider two control factors:

1. Transistor Gain – nonlinear relationship to output voltage
2. Resistance – linear relationship to output voltage

The variation in gain and resistance about their nominal values is known. Both variances are constant over the range of nominal values being considered.

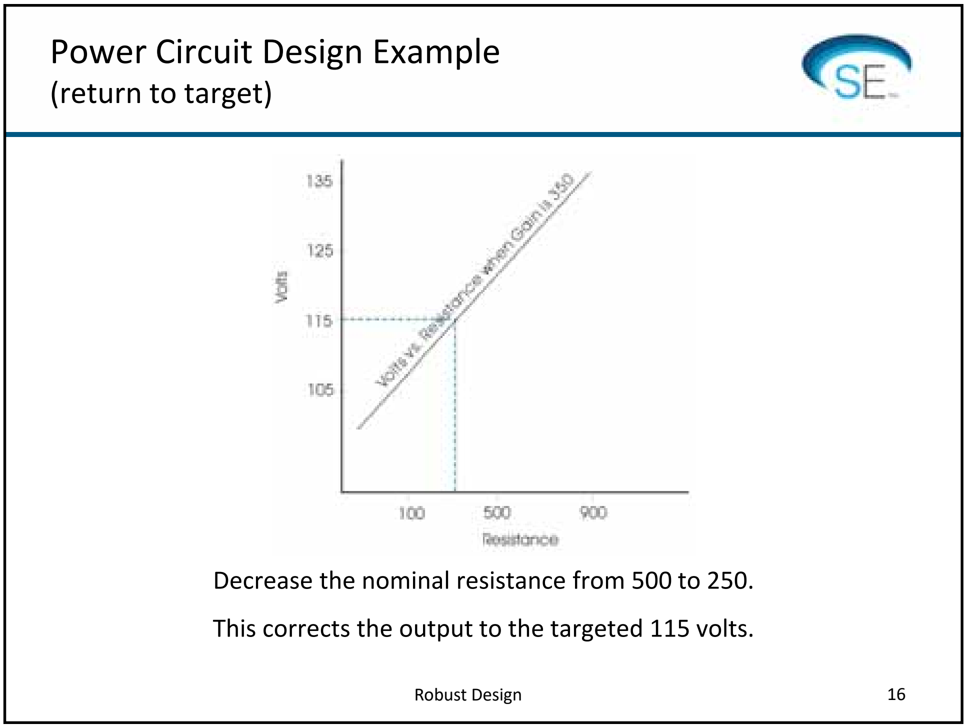




Power Circuit Design Example
(reduce variation)

Variation is reduced by using a nominal gain of 350.

That shifts the output off-target to 125 volts.



Power Circuit Design Example
(return to target)

Decrease the nominal resistance from 500 to 250.

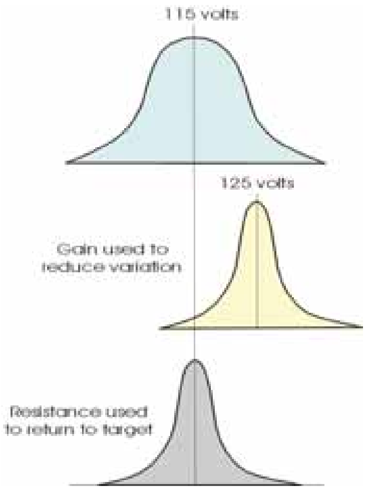
This corrects the output to the targeted 115 volts.

Power Circuit Design Example

on target with reduced variation

To illustrate the theory, the control factors were used in two steps: first to decrease variation and second to move back on target.

In practice, numerical optimization can be used to simultaneously obtain all the goals.



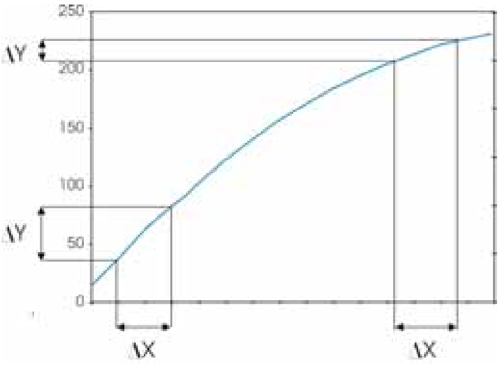
Propagation of error

Just a little mathematical explanation

Find regions where variation in the control factors transmits the least variation to the response.

$$\hat{Y} = \beta_0 + \beta_1 x_1 + \beta_{11} x_1^2$$

$$\hat{Y} = 15 + 25x_1 - 0.7x_1^2$$



The goal is to minimize the slope, which is the 1st derivative of the prediction equation.

Propagation of error

Just a brief mathematical explanation

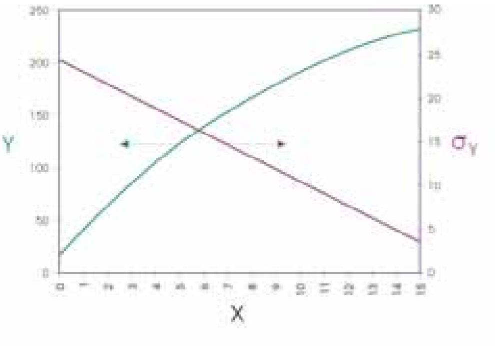
$$\hat{Y} = \beta_0 + \beta_1 x_1 + \beta_{11} x_1^2$$

$$\hat{Y} = 15 + 25x_1 - 0.7x_1^2$$

$$\sigma_{\hat{Y}}^2 = \left(\frac{\partial Y}{\partial x}\right)^2 \sigma_x^2 + \sigma_{resid}^2$$

$$\sigma_{\hat{Y}} = \sqrt{(25 - 1.4x_1)^2 \sigma_x^2 + \sigma_{resid}^2}$$

Assume $\sigma_x = 1$ and $\sigma_{resid} = 0$



As the **slope** of the relationship between X and Y decreases, the variation transmitted to Y decreases.

Propagation of error

Goal: Minimize propagated error (POE)

What is POE?

$$\sigma^2_{\hat{Y}} = \sum_i \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_{x_i}^2 + \sum_j \left(\frac{\partial f}{\partial z_j}\right)^2 \sigma_{z_j}^2 + \sigma_{resid}^2 \qquad POE = \sqrt{\sigma^2_{\hat{Y}}}$$

The amount of variation transmitted to the response (using the transfer function):

- from the lack of control of the control factors and variability from uncontrolled factors
  (you enter these standard deviations),
- plus the normal process variation
  (obtained from the ANOVA).

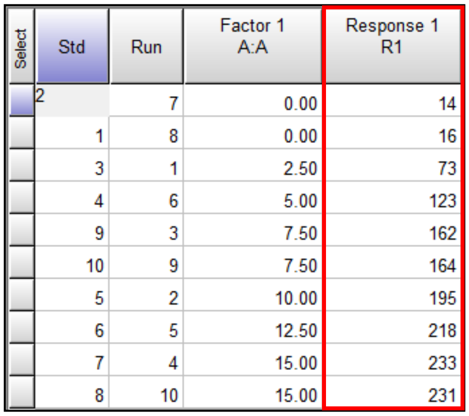
It is expressed as a standard deviation.

Propagation of error

Simple One-Factor Illustration *(page 1 of 4)*

One Factor RSM design.

- Factor A: low level = **0** and high level = **15**
- Design for a **cubic** model.
- **Sort by Factor A**



Select	Std	Run	Factor 1 A:A	Response 1 R1
	2	7	0.00	14
	1	8	0.00	16
	3	1	2.50	73
	4	6	5.00	123
	9	3	7.50	162
	10	9	7.50	164
	5	2	10.00	195
	6	5	12.50	218
	7	4	15.00	233
	8	10	15.00	231

(continued on next page)

Propagation of error

Simple One-Factor Illustration *(page 2 of 4)*

5. Compute effects and select appropriate model.

   Fitted equation (in terms of actual factor values) is:

   $$\hat{Y} = 14.98 + 25.05A - 0.71A^2$$

6. Enter the standard deviation for each factor:

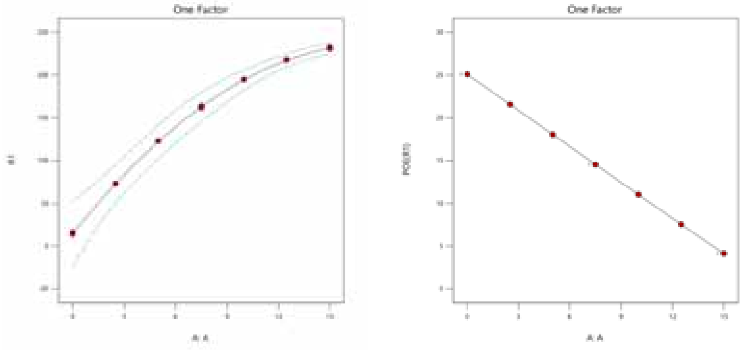
   From the Design Layout Screen - **View, Column Info Sheet** – enter **1.00** for Factor A.

   The response standard deviation is filled in automatically from the ANOVA after the analysis is completed.

Propagation of error

Simple One-Factor Illustration *(page 3 of 4)*

Analyze the response (R1) and look at the one factor plot and the propagation of error plot (from the View menu.)



$$\hat{y} = 14.98 + 25x_1 - 0.7x_1^2$$

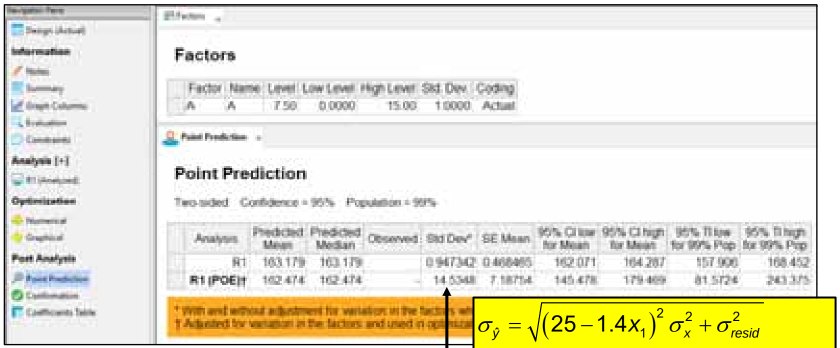
$$\sigma_{\hat{y}} = \sqrt{(25 - 1.4x_1)^2\,\sigma_x^2 + \sigma_{resid}^2}$$

assume $\sigma_x = 1$ and $\sigma_{resid} = 0.95$

Propagation of error

Simple One-Factor Illustration *(page 4 of 4)*

Point Prediction node:



Factors

Factor	Name	Level	Low Level	High Level	Std. Dev.	Coding
A	A	7.50	0.0000	15.00	1.0000	Actual

Point Prediction

Two-sided Confidence = 95% Population = 99%

Analysis	Predicted Mean	Predicted Median	Observed	Std Dev*	SE Mean	95% CI low for Mean	95% CI high for Mean	95% TI low for 99% Pop	95% TI high for 99% Pop
R1	163.179	163.179		0.947342	0.468465	162.071	164.287	157.906	168.452
R1 (POE)†	162.474	162.474	-	14.5348	7.18754	145.478	179.469	81.5724	243.375

* With and without adjustment for variation in the factors wh
† Adjusted for variation in the factors and used in optimiza

$$\sigma_{\hat{y}} = \sqrt{(25 - 1.4x_1)^2\,\sigma_x^2 + \sigma_{resid}^2}$$

$$= \sqrt{(25 - 1.4(7.5))^2\,(1)^2 + (0.95)^2}$$

$$= 14.54$$

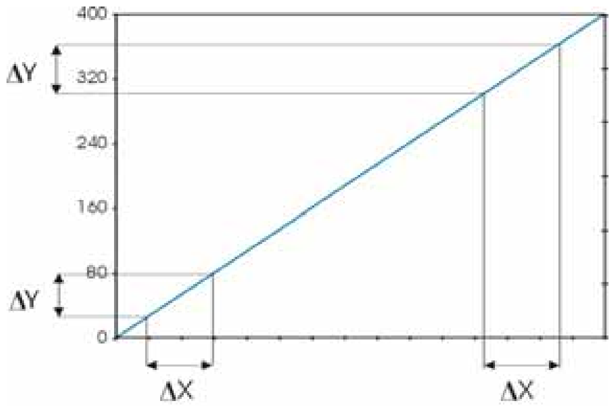
Propagation of error

Linear relationship *(page 1 of 3)*

If the response is a linear function of the independent factors, the transmitted variation is a constant.

$$\hat{y} = \beta_0 + \beta_1 x_1$$

$$\hat{y} = \mathbf{15} + \mathbf{25} x_1$$



Propagation of error

Linear relationship *(page 2 of 3)*

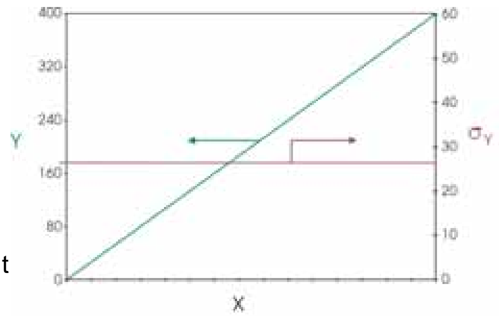
$$\hat{y} = \beta_0 + \beta_1 x_1$$

$$\hat{y} = 15 + 25 x_1$$

$$\sigma_{\hat{y}}^2 = \left(\frac{\partial Y}{\partial x}\right)^2 \sigma_x^2 + \sigma_{resid}^2$$

$$\sigma_{\hat{y}} = \sqrt{(25)^2 \sigma_x^2 + \sigma_{resid}^2} = \text{Constant}$$

assume $\sigma_x = 1$ and $\sigma_{resid} = 3.17$

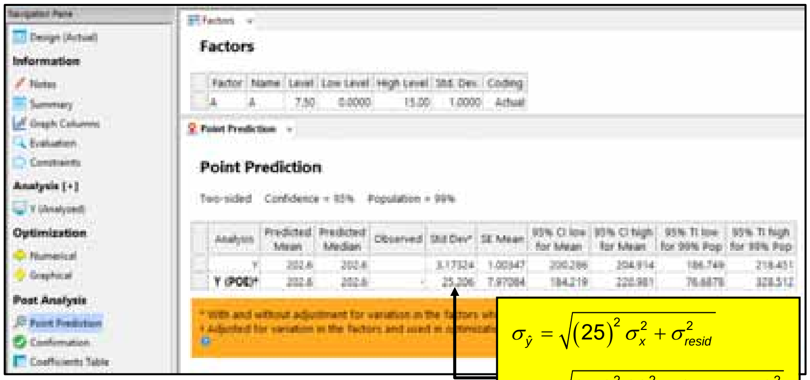


Since the slope of the relationship between X and Y is constant, the variation transmitted to the response also remains constant.

Propagation of error

Linear relationship *(page 3 of 3)*

Point Prediction node:



Factors

Factor	Name	Level	Low Level	High Level	Std. Dev.	Coding
A	A	7.50	0.0000	15.00	1.0000	Actual

Point Prediction

Two-sided Confidence = 95% Population = 99%

Analysis	Predicted Mean	Predicted Median	Observed	Std Dev*	SE Mean	95% CI low for Mean	95% CI high for Mean	95% TI low for 99% Pop	95% TI high for 99% Pop
Y	202.6	202.6		3.17324	1.00347	200.206	204.914	186.749	218.451
Y (POE)†	202.6	202.6		25.206	7.97084	184.219	220.981	76.6878	328.512

* With and without adjustment for variation in the factors wi

† Adjusted for variation in the factors and used in optimizatio

$$\sigma_{\hat{y}} = \sqrt{(25)^2\,\sigma_x^2 + \sigma_{resid}^2}$$

$$= \sqrt{(25)^2\,(1)^2 + (3.17324)^2}$$

$$= 25.206$$

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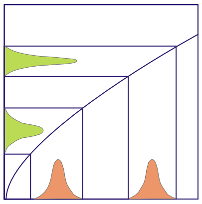
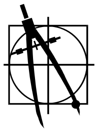


Robust RSM Simulation

Precise Machined Parts

Acme precision machine company has been having trouble holding nominal values on their new highly-automated lathe. Your job is to study the process and reduce deviations from nominal. Previous work has determined that three factors are the key influencers on the process:

Factor	**Units**	**Range**
Cutting Speed	fpm	330 - 700
Feed Rate	ipr	0.01 – 0.022
Depth of cut	inches	0.05 – 0.10



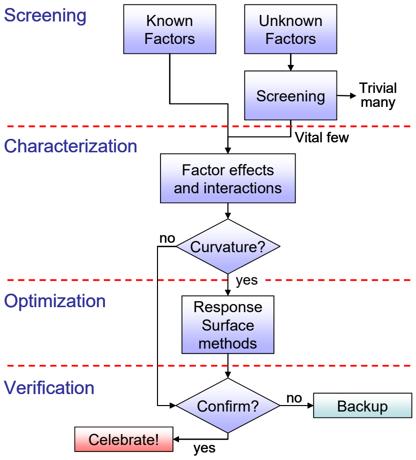
The factor levels given are the extreme values, do not exceed them.

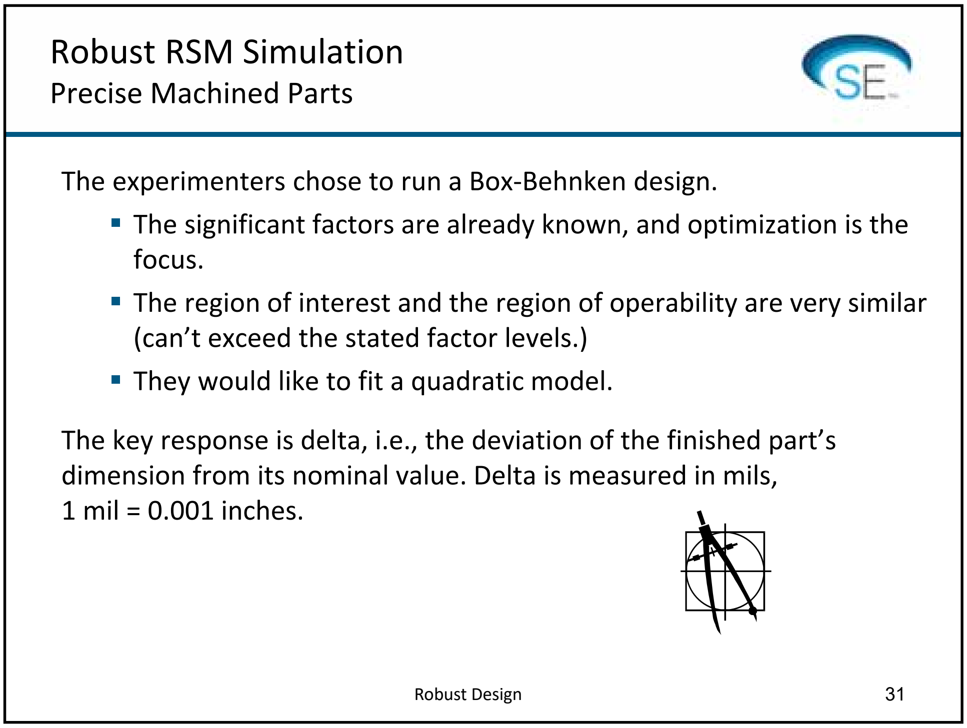
Robust RSM Simulation

Precise Machined Parts

Which design is an appropriate choice for the Lathe DOE?

What model should we design for?





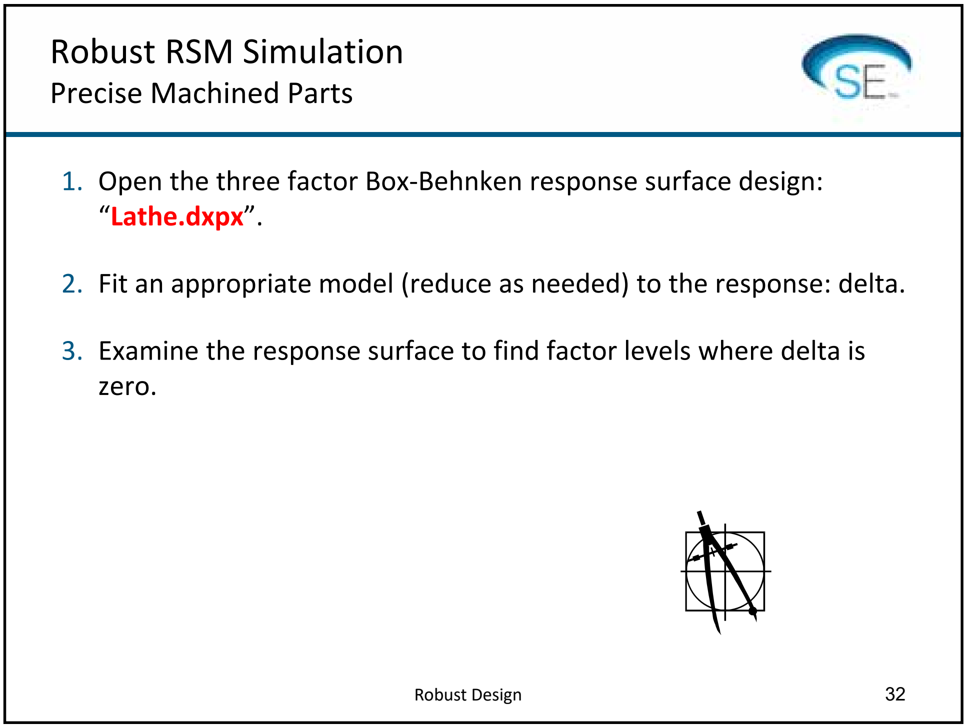
Robust RSM Simulation

Precise Machined Parts

The experimenters chose to run a Box-Behnken design.

- The significant factors are already known, and optimization is the focus.
- The region of interest and the region of operability are very similar (can't exceed the stated factor levels.)
- They would like to fit a quadratic model.

The key response is delta, i.e., the deviation of the finished part's dimension from its nominal value. Delta is measured in mils, 1 mil = 0.001 inches.



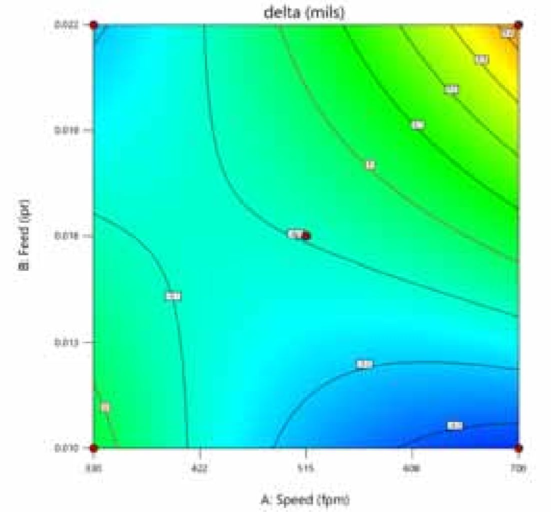
Robust RSM Simulation

Precise Machined Parts

1. Open the three factor Box-Behnken response surface design: "**Lathe.dxpx**".
2. Fit an appropriate model (reduce as needed) to the response: delta.
3. Examine the response surface to find factor levels where delta is zero.

Robust RSM Simulation

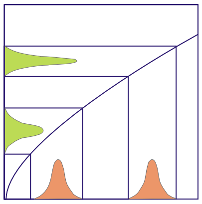
Precise Parts



Note the variety of speed and feed combinations that can produce a delta of 0.

You could also explore the other graph combinations
of speed vs. depth and feed vs. depth.

Agenda Transition



- ➢ Robust Design Concepts
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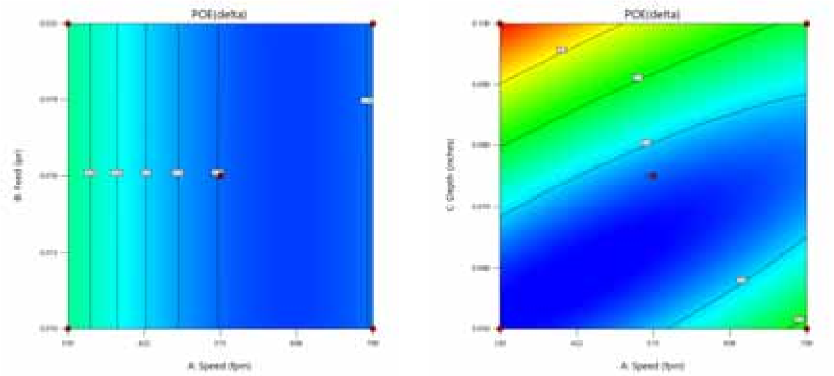
Robust RSM Simulation
Precise Parts – Add POE

1. Enter information on the expected variation of the controllable factors about their set points. From the Design node, go to "**Column Info Sheet**" and enter:

Variable	Standard Deviation
A – Speed	5 fpm
B – Feed	0.00175 ipr
C – Depth	0.0125 inches

2. Use the POE model graphs to explore the transmitted error as a function of the independent factors.

Robust RSM Simulation
Precise Parts – Add POE

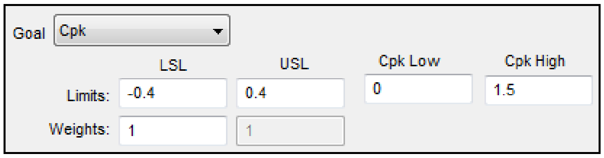


These are two of the three views of the propagated error - - where is POE minimized?

Robust RSM Simulation

Precise Parts – Optimization

1. Use numerical optimization to find factor levels near zero delta that are also robust:



Goal: Cpk

	LSL	USL	Cpk Low	Cpk High
Limits:	-0.4	0.4	0	1.5
Weights:	1	1		

2. Choose settings to operate the lathe:

Speed	517 Fpm
Feed	0.022 ipr
Depth	0.065 inches
Delta	8.1E-08 mils
POE(Delta)	0.12 mils

Since the resulting C_{pk} of **1.11** is so low, we will look into partitioning the POE to see which of the sources of variation are largest.

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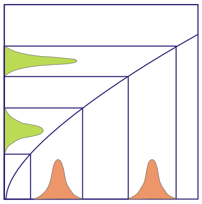


Empirical Tolerancing

Statistical detail

Predicted response:

$$\hat{y} = f\left(x_1,\dots,x_k\right) + \frac{1}{2}\sum_{i=1}^{k}\left(\frac{\partial^2 f}{\partial x_i^2}\right)\sigma_{ii}^2$$

POE (transmitted variation):

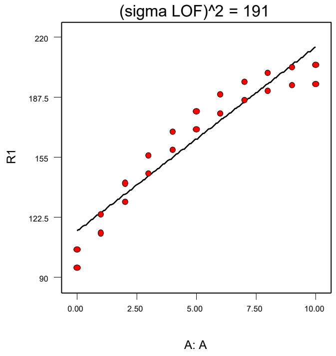
$$\sigma^2_{\hat{Y}} = \sum_{i=1}^{k}\left(\frac{\partial f}{\partial x_i}\right)^2\sigma_{ii}^2 + \frac{1}{2}\sum_{i=1}^{k}\left(\frac{\partial^2 f}{\partial x_i^2}\right)^2\sigma_{ii}^4 + \sum_{i<j}^{k}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2\sigma_{ii}^2\sigma_{jj}^2 + \sigma^2_{\text{residual}}$$

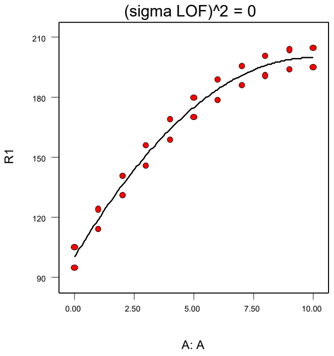
Residual MS:

$$\sigma^2_{\text{residual}} = \sigma^2_{\text{part}} + \sigma^2_{\text{measure}} + \underbrace{\sigma^2_{\text{lack-of-fit}}}_{0 \text{ if not significant}}$$

Empirical Tolerancing

Residual MS: $\sigma^2_{\text{residual}} = \sigma^2_{\text{part}} + \sigma^2_{\text{measure}} + \sigma^2_{\text{lack-of-fit}}$





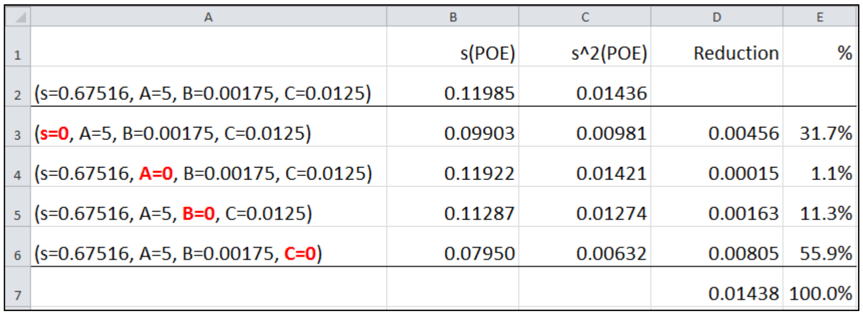
Fit an appropriate model!

Precise Machined Parts
Tolerance Analysis

Source	Sum of Squares	df	Mean Square	F Value	p-value Prob > F
Model	1.36	7	0.19	42.65	< 0.0001
A-Speed	*0.037*	*1*	*0.037*	*8.17*	*0.0189*
B-Feed	*0.15*	*1*	*0.15*	*32.42*	*0.0003*
C-Depth	*0.35*	*1*	*0.35*	*77.80*	*< 0.0001*
AB	*0.27*	*1*	*0.27*	*59.70*	*< 0.0001*
AC	*0.24*	*1*	*0.24*	*53.21*	*< 0.0001*
A^2	*0.025*	*1*	*0.025*	*5.52*	*0.0434*
C^2	*0.27*	*1*	*0.27*	*59.53*	*< 0.0001*
Residual	0.041	9	4.558E-003		
Lack of Fit	***0.019***	***5***	***3.740E-003***	***0.67***	***0.6689***
Pure Error	*0.022*	*4*	*5.582E-003*		
Cor Total	1.40	16			

Precise Machined Parts
Partitioning Variation to Inputs *(page 1 of 2)*

At the optimal point, calculate the reduction in error variance by setting one standard deviation at a time to zero:



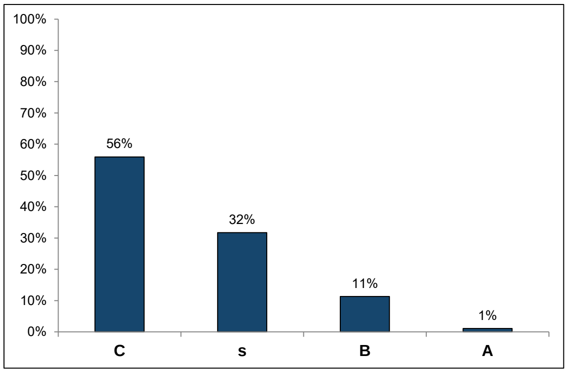
	A	B	C	D	E
1		s(POE)	s^2(POE)	Reduction	%
2	(s=0.67516, A=5, B=0.00175, C=0.0125)	0.11985	0.01436		
3	(**s=0**, A=5, B=0.00175, C=0.0125)	0.09903	0.00981	0.00456	31.7%
4	(s=0.67516, **A=0**, B=0.00175, C=0.0125)	0.11922	0.01421	0.00015	1.1%
5	(s=0.67516, A=5, **B=0**, C=0.0125)	0.11287	0.01274	0.00163	11.3%
6	(s=0.67516, A=5, B=0.00175, **C=0**)	0.07950	0.00632	0.00805	55.9%
7				0.01438	100.0%

See graph on next slide!

Precise Machined Parts

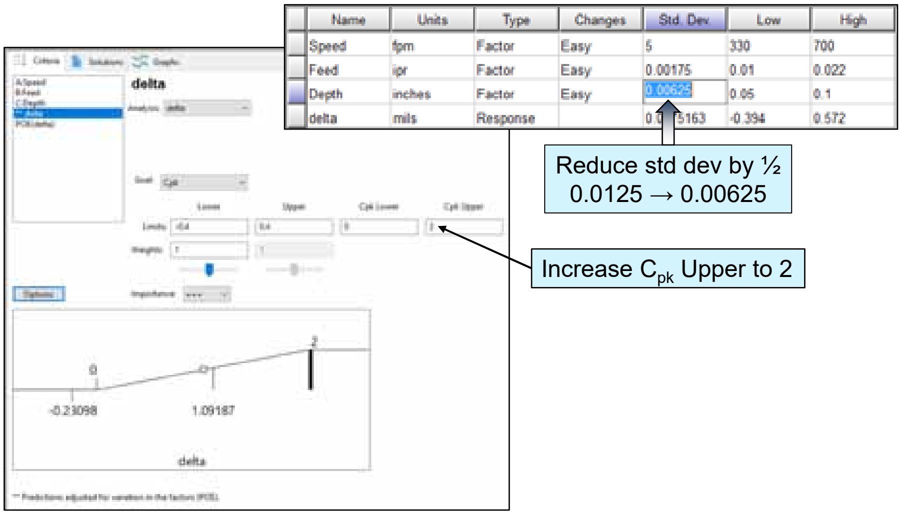
Partitioning Variation to Inputs *(page 2 of 2)*

Percent overall POE is reduced by
setting each factor standard deviation to zero



Precise Machined Parts

Reducing Variation in Depth (factor C)



Name	Units	Type	Changes	Std. Dev	Low	High
Speed	fpm	Factor	Easy	5	330	700
Feed	ipr	Factor	Easy	0.00175	0.01	0.022
Depth	inches	Factor	Easy	0.00625	0.05	0.1
delta	mils	Response		0.0[illegible]5163	-0.394	0.572

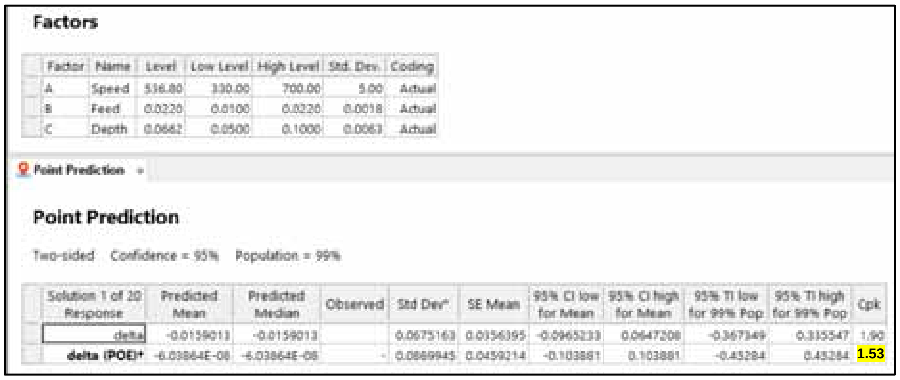
Reduce std dev by ½
0.0125 → 0.00625

Increase C_{pk} Upper to 2

Precise Machined Parts

Reducing Variation in Depth (factor C)

C_{pk} of 1.53 is a good improvement:



Factors

Factor	Name	Level	Low Level	High Level	Std. Dev.	Coding
A	Speed	536.80	330.00	700.00	5.00	Actual
B	Feed	0.0220	0.0100	0.0220	0.0018	Actual
C	Depth	0.0662	0.0500	0.1000	0.0063	Actual

Point Prediction

Two-sided Confidence = 95% Population = 99%

Solution 1 of 20 Response	Predicted Mean	Predicted Median	Observed	Std Dev*	SE Mean	95% CI low for Mean	95% CI high for Mean	95% TI low for 99% Pop	95% TI high for 99% Pop	Cpk
delta	-0.0159013	-0.0159013		0.0675163	0.0356395	-0.0965233	0.0647208	-0.367349	0.335547	1.90
delta (POE)*	-6.03864E-08	-6.03864E-08	-	0.0869945	0.0459214	-0.103881	0.103881	-0.43284	0.43284	1.53

Can sigma for factor C be reduced to 0.00625?

Precise Machined Parts

Reducing Variation in Depth (factor C)

Factor	Name	Level	Low Level	High Level	Std. Dev.	Coding
A	Speed	536.8	330.0	700.0	5.000	Actual
B	Feed	0.02200	0.01000	0.02200	0.001750	Actual
C	Depth	0.06619	0.05000	0.1000	0.006250	Actual

Solution 1 of 20 Response	Predicted Mean	Predicted Median		Std Dev*	SE Mean	95% CI low for Mean	95% CI high for Mean	95% TI low for 99% Pop	95% TI high for 99% Pop	Cpk
delta	-0.01590	-0.01590		0.06752	0.03564	-0.09652	0.06472	-0.3673	0.3355	1.896
delta (POE)	-6.039E-08	-6.039E-08	-	0.08699	0.04592	-0.1039	0.1039	**-0.4528**	**0.4528**	**1.533**

C_{pk} of 1.53 is a good improvement:

- Can sigma for factor C be reduced to 0.00625? At what cost?
- Still not six sigma, see Tolerance Intervals in relation to the specifications of -0.40 to +0.40.

References

1) Raymond H. Myers, Douglas C. Montgomery and Christine M. Anderson-Cook (2016), 4th edition, Response Surface Methodology, John Wiley and Sons, Inc, Sections 2.7-2.8 and 6.6, Chapters 1, 7 and 11.

2) George E.P. Box, William G. Hunter and J. Stuart Hunter (2005), *Statistics for Experimenters*, 2nd edition John Wiley, Chapter 13.

3) Steven DE Gryze, Ivan Langhans and Martina Vandebroek, “Using the Intervals for Prediction: A Tutorial on Tolerance Intervals for Ordinary Least-Squares Regression”, Chemometrics and Intelligent Laboratory Systems, 87 (2007) 147 – 154.

4) Gerald J. Hahn and William Q. Meeker, Statistical Intervals; A Guide for Practitioners, (1991) John Wiley and Sons, Inc.

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