Graphical Selection of Effects in General Factorials

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Existing Graphical Methods

Existing methods for factorials with more than two levels either require that:

- the entire design be split into single degree of freedom effects with equal variance (so that the usual half-normal plot will work), or
- all effects have the same degrees of freedom *(which can be more than 1)*.

Graphical Selection of Effects in General Factorials

Our context is a general factorial model with k factors; each factor has two or more levels.

We seek an extension to the Daniel half-normal plot that:

1. will be equivalent to the usual plot for balanced two level designs,
2. will permit factors with more than two levels,
3. Will do sensible things for unbalanced data.

Outline

- **Present a general method to estimate, plot and select factorial effects.**
  - Demonstrate equivalency of new method with Daniel's half-normal plot of effects for two-level factorials.
- Demonstrate the general method:
  - Two replicates of a 3×2 factorial.
  - Two replicates of a 3×2×2 factorial.
  - Single replicate of a 3×4×4 factorial.
- Summary

Use Effect SS

Daniel's plot uses estimated effects, but we base our plot on effect SS because:

- Estimated effects are somewhat arbitrary with two or more df, but the SS for a term can be well defined.
- If we knew the error variance, we could test the terms using their SS via a χ^2 test.

We do not know the error variance, but suppose that we have a provisional value for the error variance; call it $\tilde{\sigma}^2$.

Chi-Squared Statistic

To test

H_0: model term i has no effect on the mean against

H_A: model term i does effect on the mean

we see if the SS for an effect is larger than expected given the provision estimate of error variance?

$$\chi^2 = \frac{SS_{effect}}{\tilde{\sigma}^2} = \frac{df\left(MS_{effect}\right)}{\tilde{\sigma}^2}$$

Provisional p-value

Using this provisional $\tilde{\sigma}$, we can compute provisional p-values:

$$\tilde{p}_i = G\left(\frac{SS_i}{\tilde{\sigma}^2}, df_i\right)$$

where $G(\chi,\nu)$ is the upper tail for a Chi-squared random variable with ν degrees of freedom.

We use $\tilde{p}_i$ to indicate a provisional p-value value based on our provisional estimate of error variance.

Normal Effect

We can also express this provisional p-value as a provisional z-score from the half-normal distribution:

$$\tilde{z}_i = \Phi^{-1}\left(1 - \frac{\tilde{p}_i}{2}\right)$$

where Φ is the cumulative distribution function of a standard normal.

We call this the “Normal Effect”; it is what we plot for terms in factorials when factors have any number of levels.

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Equivalency with Daniel Plots

Balanced two-series Designs *(page 1 of 2)*

In the balanced two-series design:

- All effects e_i have df=1
- Effect e_i has $SS_i = (e_i^2)(2^k/4)$

Thus

$$\tilde{p}_i = G\left(\frac{SS_i}{\tilde{\sigma}^2}, df_i\right) = G\left(\frac{e_i^2 2^k}{4\tilde{\sigma}^2}, df_i\right) = 2\left(1 - \Phi\left(\frac{\left(2^{k/2}|e_i|\right)}{2\tilde{\sigma}}\right)\right)$$

and

$$\tilde{z}_i = \Phi^{-1}\left(1 - \frac{\tilde{p}_i}{2}\right) = \frac{\left(2^{k/2}|e_i|\right)}{2\tilde{\sigma}} \quad \therefore \quad \tilde{z}_i \propto |e_i|$$

Equivalency with Daniel Plots

Balanced two-series Designs *(page 2 of 2)*

So on the balanced two-series design:

- The normal effects are proportional to the ordinary effects.
- In fact we could use a different $\tilde{\sigma}^2$ and still get the same relative plot *(with a rescaled horizontal axis)*.
- This equivalence to Daniel's plot in the two-series motivates using chi-square and provisional error instead of the standard F.

Two-Level Factorial Example

2^4 full factorial

Factor	Name	Units	Levels
A	Temperature	deg C	2
B	Pressure	psig	2
C	Concentration	percent	2
D	Stir Rate	rpm	2

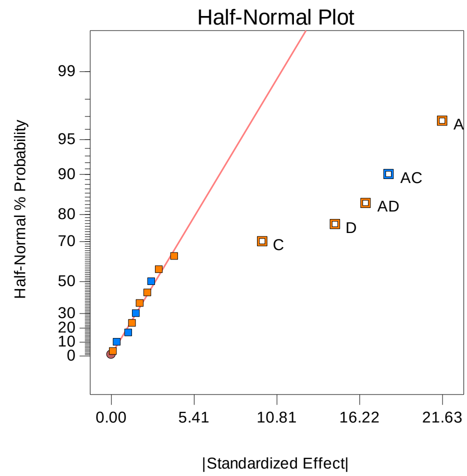
Example 6-2; page 228:

Douglas C. Montgomery (2005), 6th edition, *Design and Analysis of Experiments*, John Wiley and Sons, Inc.

Two-Level Example
2^4 full factorial (Usual Effects)

Design-Expert® Software
Filtration Rate

Shapiro-Wilk test
W-value = 0.974
p-value = 0.927
A: Temperature
B: Pressure
C: Concentration
D: Stir Rate
Positive Effects
Negative Effects



Graphical Selection

Two-Level Example
2^4 full factorial (Normal Effects)

Design-Expert® Software
Filtration Rate

Shapiro-Wilk test
W-value = 0.974
p-value = 0.927
A: Temperature
B: Pressure
C: Concentration
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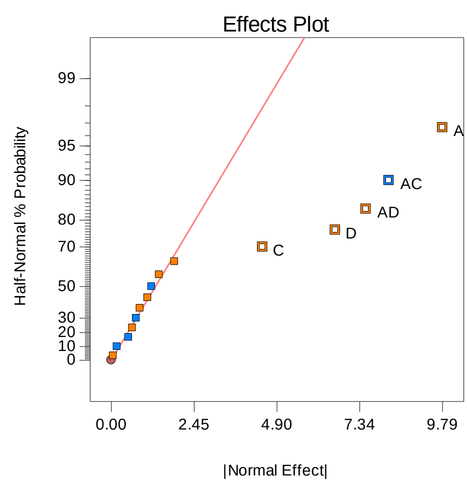
Positive Effects
Negative Effects



Graphical Selection

Nagging Questions

This looks good, but the obvious questions are:

1. Where do we get $\tilde{\sigma}^2$?
2. How do we define the SS for an effect in unbalanced data?

Estimating Provisional Error

1. Where do we get $\tilde{\sigma}^2$?
   a. Before any model terms are selected:
      - The SS for terms selected via forward selection using a Bonferroni correction [α/(# effects)] are removed from error SS. *If no terms are selected the error SS is the corrected total SS.*
      - $\tilde{\sigma}_i^2$ is estimated by error SS minus the SS for the effect being tested divided by ($df_{error} - df_{effect}$).

   b. After any model terms are selected:
      - The SS for unselected effects are pooled with pure error SS (if present) to estimate error, $\tilde{\sigma}^2$.

Unbalanced Designs

When data are balanced (more generally, model terms are orthogonal):

- The estimated effects for terms and the sums of squares for terms do not depend on which terms are in the model.

When data are unbalanced, both effects and sums of squares depend on which other terms are present:

- This means that we cannot refer to the sum of squares for a term, but only to the sum of squares for a term in a particular model.

Effect Sum of Squares

2. How do we define the SS for an effect in unbalanced data?

   We need to be precise. Suppose that the model at present contains a set of terms *(this could be only the constant)*.

   - The SS for a term in the model is the SS for removing that term from the model.
   - The SS for a term not in the model is the SS for adding that term to the model.

   These are effectively the sums of squares used in stepwise regression.

A Dynamic Probability Plot

Estimating $\tilde{\sigma}^2$:

If we add or remove a term to the model, we need to recompute $\tilde{\sigma}^2$ and therefore recompute the "Normal Effects" $\tilde{z}_i$.

This makes for a somewhat dynamic probability plot.

Estimating Provisional Error

2^4 full factorial *(page 1 of 2)*

Mean only model:

Source	Sum of Squares	df
Unselected Effects	*5730.94*	*15*
Pure Error	*0.0*	*0*
Corrected Total	*5730.94*	*15*

No terms picked by forward selection with a Bonferroni corrected alpha (*0.05/15 = 0.00333*):

Estimating Provisional Error

2^4 full factorial *(page 2 of 2)*

$$\tilde{\sigma}_i^2 = \frac{5730.94 - SS_i}{15 - df_i}$$

$$\tilde{\sigma}_A^2 = \frac{5730.94 - 1870.56}{15 - 1} = 275.74$$

$$\tilde{\sigma}_B^2 = \frac{5730.94 - 39.06}{15 - 1} = 406.56$$

$$\vdots$$

$$\tilde{\sigma}_{ABCD}^2 = \frac{5730.94 - 7.56}{15 - 1} = 408.81$$

Provisional p-value

2^4 full factorial

Using the provisional $\tilde{\sigma}$ compute provisional p-values:

$$\tilde{p}_i = G\left(\frac{SS_i}{\tilde{\sigma}_i^2}, df_i\right)$$

$$\tilde{p}_A = G\left(\frac{1870.56}{275.74}, 1\right) = 0.0092$$

$$\tilde{p}_B = G\left(\frac{39.0625}{406.56}, 1\right) = 0.7566$$

$$\vdots$$

$$\tilde{p}_{ABCD} = G\left(\frac{7.5625}{408.81}, 1\right) = 0.8918$$

Normal Effect
2^4 full factorial

Express the provisional p-values as provisional z-scores from the half-normal distribution:

$$\tilde{z}_i = \phi^{-1}\left(1 - \frac{\tilde{p}_i}{2}\right)$$

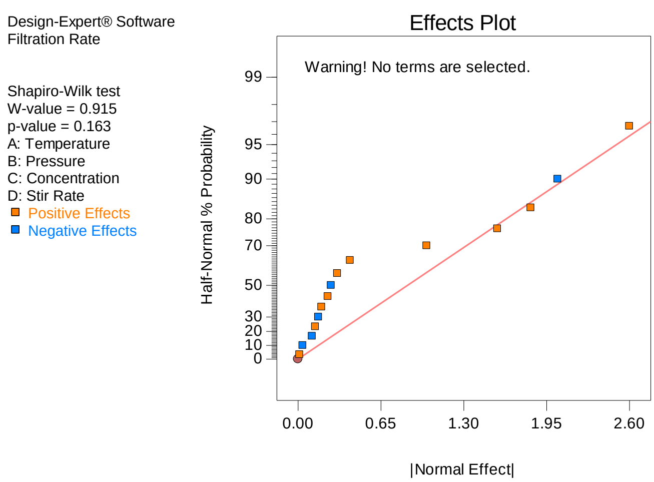
$$\tilde{z}_A = \phi^{-1}\left(1 - \frac{0.0092}{2}\right) = 2.605$$

$$\tilde{z}_B = \phi^{-1}\left(1 - \frac{0.7566}{2}\right) = 0.310$$

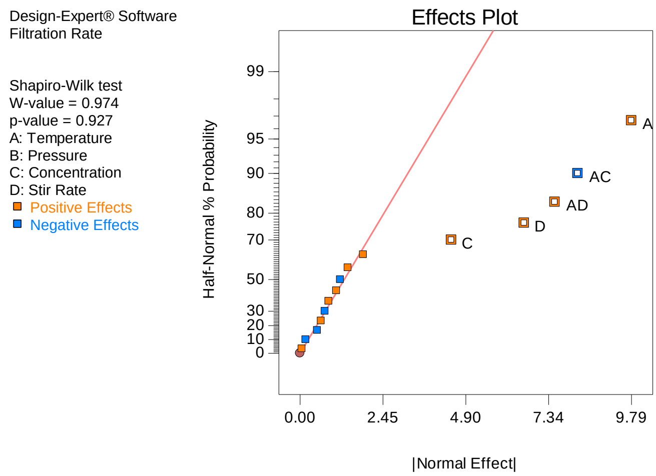
$$\vdots$$

$$\tilde{z}_{ABCD} = \phi^{-1}\left(1 - \frac{0.8918}{2}\right) = 0.136$$

Two-Level Example
Normal Effects – Nothing selected



Two-Level Example
Normal Effects – After selection



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Estimating Provisional Error
2^4 full factorial

A, C, D, AC and AD selected:

Source	Sum of Squares	df
Unselected Effects	*195.13*	*10*
Pure Error	*0.0*	*0*
Corrected Total	*5730.94*	*15*

$$\tilde{\sigma}^2 = \frac{195.13}{10} = 19.513$$

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Provisional p-value

2^4 full factorial

Using the provisional $\tilde{\sigma}$ compute provisional p-values:

$$\tilde{p}_i = G\left(\frac{SS_i}{\tilde{\sigma}^2}, df_i\right)$$

$$\tilde{p}_A = G\left(\frac{1870.56}{19.513}, 1\right) = 1.230E\text{-}22$$

$$\tilde{p}_B = G\left(\frac{39.0625}{19.513}, 1\right) = 1.571E\text{-}01$$

$$\vdots$$

$$\tilde{p}_{ABCD} = G\left(\frac{7.5625}{19.513}, 1\right) = 5.336E\text{-}01$$

Normal Effect

2^4 full factorial

Express the provisional p-values as provisional z-scores from the half-normal distribution:

$$\tilde{z}_i = \phi^{-1}\left(1 - \frac{\tilde{p}_i}{2}\right)$$

$$\tilde{z}_A = \phi^{-1}\left(1 - \frac{1.2300\text{E-}22}{2}\right) = 9.791$$

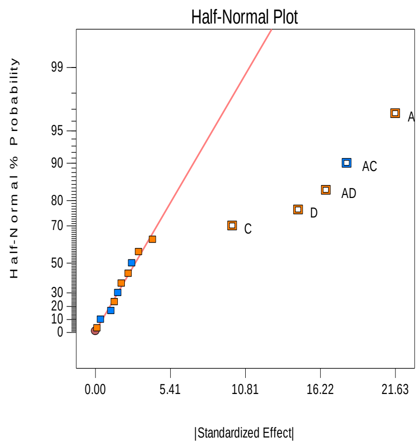
$$\tilde{z}_B = \phi^{-1}\left(1 - \frac{0.1571}{2}\right) = 1.415$$

$$\vdots$$

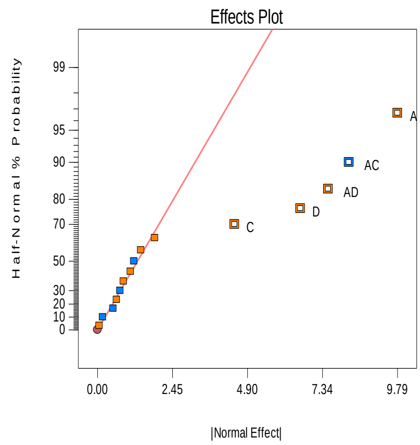
$$\tilde{z}_{ABCD} = \phi^{-1}\left(1 - \frac{0.5336}{2}\right) = 0.622$$

Two-Level Example
Equivalency!

Usual Effects



Normal Effects



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  - Summary

General Factorial Example

Two replicates of a 3×2 factorial

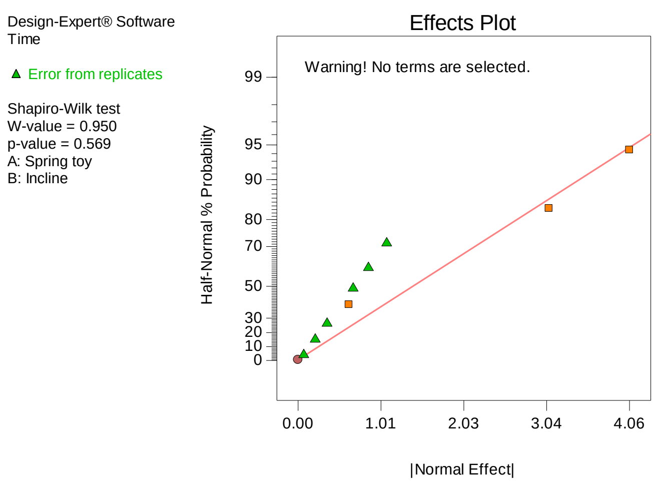
Factor	Name	Units	Type	Levels
A	Spring Toy		Categoric	3
B	Incline		Categoric	2

Table 7-1; page 136:

Mark J. Anderson and Patrick J. Whitcomb (2007), 2nd edition, *DOE Simplified – Practical Tools for Effective Experimentation*, Productivity, Inc.

General Factorial Example

Two replicates of a 3×2 factorial



Estimating Provisional Error

Two replicates of a 3×2 factorial *(page 1 of 2)*

Mean only model:

Source	Sum of Squares	df
Effects	*7.73*	*5*
Pure Error	*0.85*	*6*
Corrected Total	*8.58*	*23*

A is picked by forward selection with a Bonferroni corrected alpha (*0.05/3 = 0.0167*):

Source	Sum of Squares	df
Bonferroni terms (A)	*5.90*	*2*

Estimating Provisional Error

Two replicates of a 3×2 factorial *(page 2 of 2)*

Bonferroni terms: A:

Source	*Sum of Squares*	*df*
Cor Total	*8.58*	*11*
Bonferroni terms	*5.90*	*2*
Provisional error	***2.68***	***9***

$$\tilde{\sigma}_i^2 = \frac{2.68}{9} = 0.30 \text{ for A}$$

$$\tilde{\sigma}_i^2 = \frac{2.68 - SS_i}{9 - df_i} \text{ for B and AB}$$

Provisional p-value

Two replicates of a 3×2 factorial

Using the provisional $\tilde{\sigma}$ compute provisional p-values:

$$\tilde{p}_i = G\left(\frac{SS_i}{\tilde{\sigma}_i^2}, df_i\right)$$

$$\tilde{p}_A = G\left(\frac{5.90}{0.30}, 2\right) = 4.946E\text{-}05$$

$$\tilde{p}_B = G\left(\frac{0.12}{0.32}, 1\right) = 5.331E\text{-}01$$

$$\tilde{p}_{AB} = G\left(\frac{1.71}{0.14}, 2\right) = 2.124E\text{-}03$$

Normal Effect

Two replicates of a 3×2 factorial

Express the provisional p-values as provisional z-scores from the half-normal distribution:

$$\tilde{z}_i = \phi^{-1}\left(1 - \frac{\tilde{p}_i}{2}\right)$$

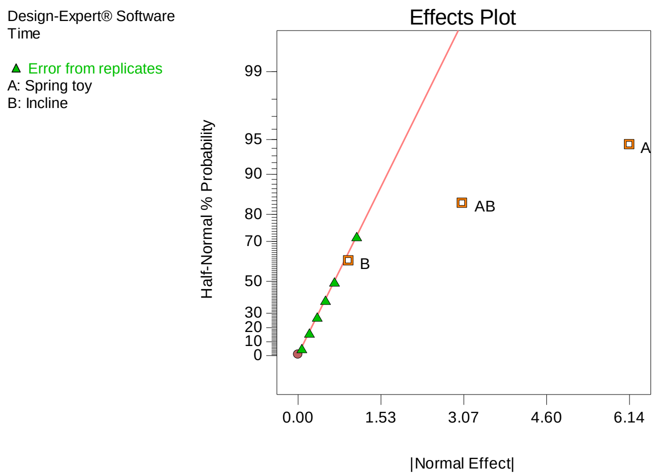
$$\tilde{z}_A = \phi^{-1}\left(1 - \frac{4.946E\text{-}05}{2}\right) = 4.058$$

$$\tilde{z}_B = \phi^{-1}\left(1 - \frac{5.331E\text{-}01}{2}\right) = 0.623$$

$$\tilde{z}_{ABC} = \phi^{-1}\left(1 - \frac{2.124E-03}{2}\right) = 3.073$$

General Factorial Example

Two replicates of a 3×2 factorial



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Estimating Provisional Error

Two replicates of a 3×2 factorial

Model terms: A, B and **AB** *(B added for hierarchy)*

Source	***Sum of Squares***	***df***
Cor Total	*8.58*	*11*
Model	*7.73*	*5*
Provisional error	***0.85***	***6***

$$\tilde{\sigma}^2 = \frac{0.85}{6} = 0.14$$

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Provisional p-value

Two replicates of a 3×2 factorial

Using the provisional $\tilde{\sigma}$ compute provisional p-values:

$$\tilde{p}_i = G\left(\frac{SS_i}{\tilde{\sigma}^2}, df_i\right)$$

$$\tilde{p}_A = G\left(\frac{5.90}{0.14}, 2\right) = 8.364E-10$$

$$\tilde{p}_B = G\left(\frac{0.12}{0.14}, 1\right) = 3.486E-01$$

$$\tilde{p}_{AB} = G\left(\frac{1.71}{0.14}, 2\right) = 2.363E-03$$

Normal Effect

Two replicates of a 3×2 factorial

Express the provisional p-values as provisional z-scores from the half-normal distribution:

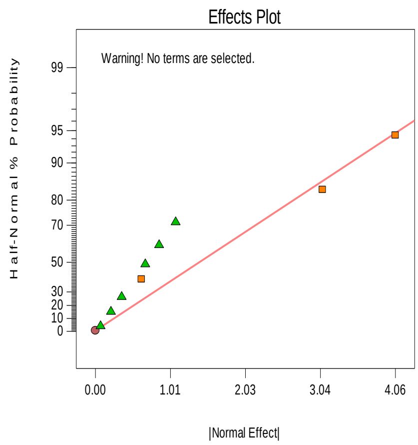
$$\tilde{z}_i = \phi^{-1}\left(1 - \frac{\tilde{p}_i}{2}\right)$$

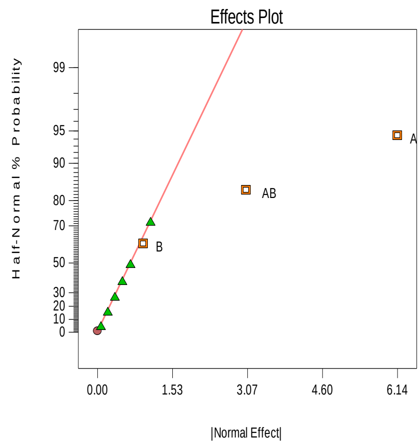
$$\tilde{z}_A = \phi^{-1}\left(1 - \frac{8.364E-10}{2}\right) = 6.137$$

$$\tilde{z}_B = \phi^{-1}\left(1 - \frac{3.486E-01}{2}\right) = 0.937$$

$$\tilde{z}_{AB} = \phi^{-1}\left(1 - \frac{2.363E-03}{2}\right) = 3.041$$

General Factorial Example
Two replicates of a 3×2 factorial





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  - **Two replicates of a 3×2×2 factorial.**
  - Single replicate of a 3×4×4 factorial.
  - Summary

General Factorial Example

Two replicates of a 3×2×2 factorial

Factor	Name	Units	Type	Levels
A	carbonation	%	Categoric	3
B	pressure	psi	Categoric	2
C	line speed	bpm	Categoric	2

Example 5-3; page 184:

Douglas C. Montgomery (2005), 6th edition, *Design and Analysis of Experiments*, John Wiley and Sons, Inc.

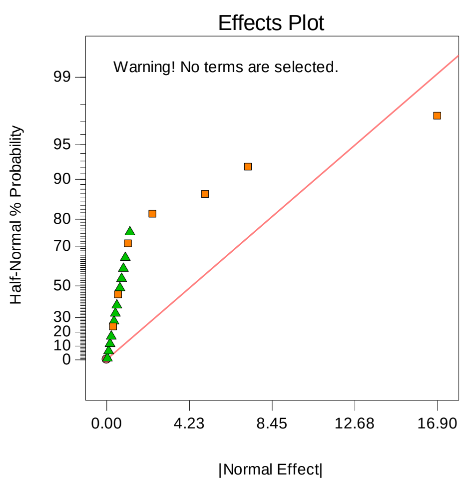
General Factorial Example

Two replicates of a 3×2×2 factorial

Design-Expert® Software
fill height

▲ Error from replicates

Shapiro-Wilk test
W-value = 0.795
p-value = 0.036
A: carbonation
B: pressure
C: line speed



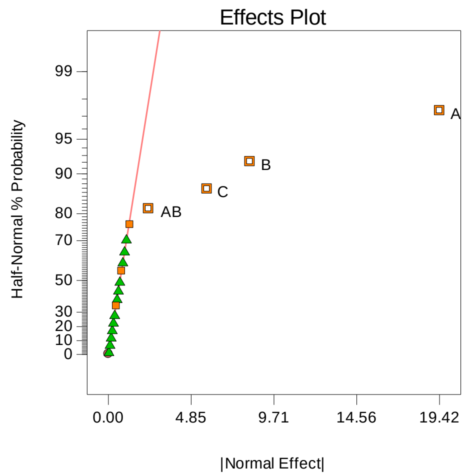
General Factorial Example

Two replicates of a 3×2×2 factorial

Design-Expert® Software
fill height

▲ Error from replicates

Shapiro-Wilk test
W-value = 0.814
p-value = 0.148
A: carbonation
B: pressure
C: line speed



Graphical Selection

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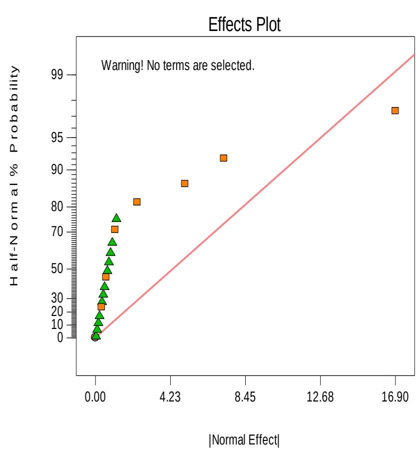
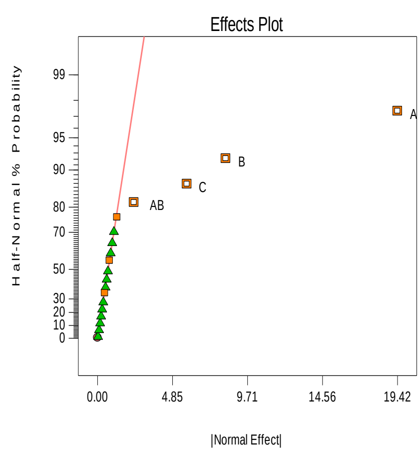
General Factorial Example

Two replicates of a 3×2×2 factorial





Graphical Selection

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General Factorial Example
Single replicate of a 3×4×4 factorial

Factor	Name	Units	Type	Levels
A	Height	inches	Categoric	3
B	Interval	weeks	Categoric	4
C	fertilizer	hp/acre	Categoric	4

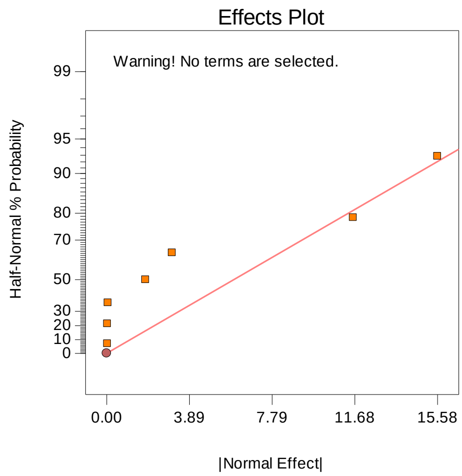
Problem 8.6; page 201:

Gary W. Oehlert (2000), *A First Course in Design and Analysis of Experiments*, W. H. Freeman and Company.

General Factorial Example
Single replicate of a 3×4×4 factorial

Design-Expert® Software
Dry yield

Shapiro-Wilk test
W-value = 0.651
p-value = 0.001
A: Height
B: Interval
C: fertilizer



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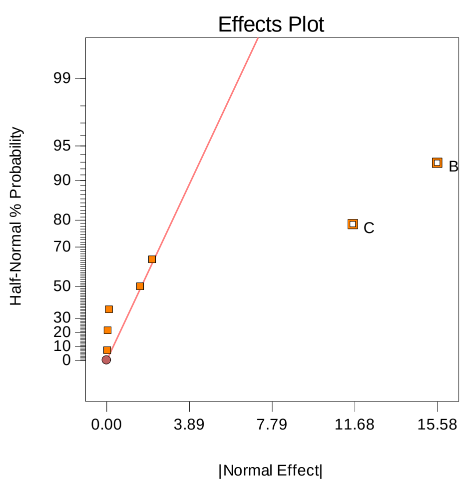
General Factorial Example
Single replicate of a 3×4×4 factorial

Design-Expert® Software
Dry yield

Shapiro-Wilk test
W-value = 0.872
p-value = 0.275
A: Height
B: Interval
C: fertilizer



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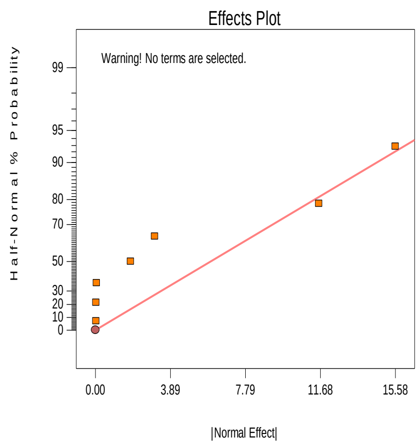
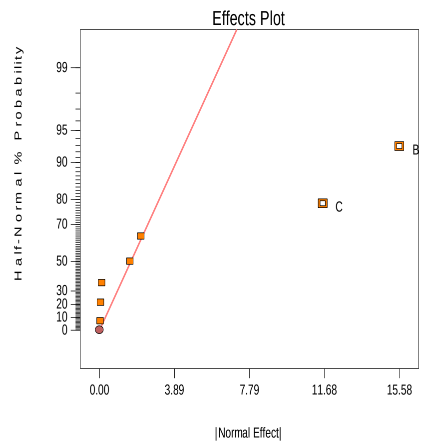
General Factorial Example

Single replicate of a 3×4×4 factorial





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Graphical Selection of Effects in General Factorials

Our context is a general factorial model with k factors; each factor has two or more levels.

We seek an extension to the Daniel half-normal plot of effects that will be equivalent to the usual plot when all factors have two levels, but will permit factors with more than two levels.

Mission Accomplished!

Graphical Selection of Effects in General Factorials

References:

[1] Cuthbert Daniel (1959). "*Use of half-normal plots in interpreting factorial two-level experiments*", Technometrics 1, 311–341.

[2] Patrick Whitcomb and Kinley Larntz, (1992). "*The role of pure error on normal probability plots*", ASQC Technical Conference Transactions,1 223-1229 American Society for Quality Control: Milwaukee.

[3] Douglas C. Montgomery (2005), 6th edition, *Design and Analysis of Experiments*, John Wiley and Sons, Inc.

[4] Mark J. Anderson and Patrick J. Whitcomb (2007), 2nd edition, *DOE Simplified – Practical Tools for Effective Experimentation*, Productivity, Inc.

[5] Gary W. Oehlert (2000), *A First Course in Design and Analysis of Experiments*, W. H. Freeman and Company.

[6] Cuthbert Daniel (1976), *Applications of statistics to industrial experimentation*, John Wiley and Sons, Inc.

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Thank you for your attention!

2007 Fall Technical Conference